A Nonlinear Damping Model for a Displacement Control System in Seismic Isolation

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Abstract

In (1) a displacement control system is described for the seismic isolation of buildings that produces a nonlinear damping force of a complicated nature.

It is proposed to approximate such a damping force by a combination of three different types of damping: linear viscous damping, constant Coulomb friction, and linear Coulomb friction. The term linear Coulomb friction is used for a damping that increases linearly with the displacement but is in phase with the velocity. This approximation leads to a nonlinear differential equation that can be solved exactly for certain cases.

The exact solution to the steady-state vibration of a single-degree-of-freedom system governed by this differential equation is given and two possibilities for an equivalent linearization of the nonlinear problem are discussed and compared with the exact solution.

The equivalent linearization of the proposed general damping model are shown to give excellent results for medium levels of damping. For large levels of damping compromises in trying to model the exact solution have to be accepted that will largely depend on the particular method of linearization used.

For most practical purposes the methods discussed offer a very simple and efficient way to define an equivalent viscous damping ratio that could be used in a standard linear analysis to quite accurately predict the response of a system which involves a rather complicated, nonlinear type of damping.

Introduction

Base isolation has long been suggested as an effective approach for aseismic design of structures. It involves, essentially, a compromise between the forces induced within a superstructure and the relative displacement requirements imposed at the base. In (1) a frictionally damped system has been considered for controlling the relative displacement at the base of an isolated structure and for preventing catastrophic collapse in the event of an earthquake which is greater in intensity than the isolation system was designed for. The downward deflection of this system associated with the relative displacements at the base of a structure can be utilized to apply a frictional damping force which steadily increases with increasing relative displacements at the base.
The total damping force of such a system can thus be approximated by a combination of three effects: a linear viscous damping force associated with the isolation system, a constant frictional force (constant Coulomb damping) and a frictional force which increases with the displacement (linear Coulomb damping) associated with a sliding friction system.

Viscous, constant Coulomb and linear Coulomb damping

The damping force described above can mathematically be described by the following expression

$$\begin{aligned} F_d &= -c\dot{x} - sign(\dot{x})\left[F + \zeta k|x|\right] \\ &= -F_v - F_c - F_l \end{aligned} \tag{1}$$

where

F_v = viscous damping force — c = viscous damping constant

F_c = constant Coulomb friction force — F = frictional force at zero deflection

F_l = linear Coulomb friction force — ζk = slope of the linearly increasing damping force

The relation between the Coulomb frictional forces and the corresponding displacement resulting from this expression is given in Fig. 1 and experimentally recorded curves to be approximated by this model are shown in Fig. 3. These curves were taken from (1).

When this model is incorporated into the dynamic analysis of a single-degree-of-freedom system the differential equation of motion is as follows

$$m\ddot{x} + c\dot{x} + kx + sign(\dot{x})\left[F + \zeta k|x|\right] = f(t) \tag{2a}$$

or

$$\ddot{x} + 2\xi\omega_0\dot{x} + \omega_0^2(1 \pm \zeta)x \pm \omega_0^2 x_c = \omega_0^2 \frac{f(t)}{k} \tag{2b}$$

where

m = mass of the system

k = spring stiffness of the system

ζ = coefficient of linear Coulomb friction

(ζk is the slope of the linearly increasing friction)

F = absolute value of constant Coulomb friction force

$x_c = \frac{F}{k}$

ξ = coefficient of viscous friction

$\omega_0 = \left\{\frac{k}{m}\right\}^{1/2}$ natural frequency

A physical representation of this equation of motion is shown in Fig. 2.

Steady State Response

The problem defined by equation (2) is piecewise linear locally, although nonlinear globally. Hence, an exact solution can be attempted by considering each linear piece of the motion separately and forming the appropriate continuity conditions.
The basic assumptions underlying a steady state analysis are: 1) that a "steady state" can actually be achieved by the system; 2) that the system will move at the frequency of the exciting force and 3) that the motion will be symmetrical for each half-cycle. Using these assumptions Eqn. (2) can be solved for a particular exciting force and the general solution can be adjusted by the appropriate boundary conditions. The forcing function chosen in this analysis was

$$f(t) = P\cos(\omega t+\phi) \qquad (3)$$

where ϕ is some unknown phase angle

Hence we have the following differential equation governing this type of motion

$$\ddot{x}+2\xi\omega_0\dot{x}+\omega_0^2(1\pm\zeta)x\pm\omega_0^2x_c = \omega_0^2 a\cos(\omega t+\phi)$$

where

$a = \frac{P}{k}$

and the signs have to be adjusted according to Eqn. (2a).

Considering only the motion for that half-cycle where the velocity is negative (i.e. from the maximum displacement x_o to the minimum displacement $-x_0$) and assuming the motion to by symmetrical for the other half-cycle the following complete solution is obtained

$$x_1(t) = e^{-\xi\omega_0 t}\left[C_1 \sin\omega_d^- t + C_2 \cos\omega_d^- t\right] + \frac{a}{q^-}\cos\left[\omega t+\phi-\theta^-\right] + x_c^- \qquad (4a)$$

for $x_0 \leqslant x_1(t) \leqslant 0$ and $\dot{x}_1(t) \leqslant 0$

where

$$q^- = \left\{\left[1-\zeta-\beta^2\right]^2 + \left[2\beta\xi\right]^2\right\}^{1/2} \qquad \theta^- = \tan^{-1}\frac{2\xi\beta}{1-\zeta-\beta^2}$$

$$\omega_d^- = \omega_0\sqrt{1-\zeta-\xi^2} \qquad x_c^- = \frac{1}{1-\zeta}\,x_c$$

$$\beta = \frac{\omega}{\omega_0} \quad \textit{frequency ratio}$$

and

$$x_2(t) = e^{-\xi\omega_0(t-t_1)}\left[C_1 \sin\omega_d^+(t-t_1) + C_2 \cos\omega_d^+(t-t_1)\right] + \frac{a}{q^+}\cos\left[\omega t+\phi-\theta^+\right] + x_c^+ \qquad (4b)$$

for $0 \leqslant x_2(t) \leqslant -x_0$ and $\dot{x}_2(t) \leqslant 0$

where

$$q^+ = \left\{\left[1+\zeta-\beta^2\right]^2 + \left[2\beta\xi\right]^2\right\}^{1/2} \qquad \theta^+ = \tan^{-1}\frac{2\xi\beta}{1+\zeta-\beta^2}$$

$$\omega_d^+ = \omega_0\sqrt{1+\zeta-\xi^2} \qquad x_c^+ = \frac{1}{1+\zeta}\,x_c$$

The following initial conditions exist for Eqns. (4)

$$x_1(t=0) = x_0 \qquad \dot{x}_1(t=0) = 0$$

$$x_2(t=t_1) = 0 \qquad \dot{x}_2(t=t_1) = v_0$$

These can be used to determine the constants C_1 through C_4

$$C_1 = \frac{\xi\omega_0}{\omega_d^-}\left\{x_0 - x_c^- - \frac{a}{q^-}\cos(\phi-\theta^-)\right\} + \frac{a}{q^-}\,\frac{\omega}{\omega_d^-}\sin(\phi-\theta^-)$$

$$C_2 = x_0 - x_c^- - \frac{a}{q^-}\cos(\phi-\theta^-)$$

$$C_3 = \frac{v_0}{\omega_d^+} - \frac{\xi\omega_0}{\omega_d^+}\,x_c^+ + \frac{a}{q^+}\,\frac{1}{\omega_d^+}\left\{-\xi\omega_0\cos(\omega t_1+\phi-\theta^+) + \omega\sin(\omega t_1+\phi-\theta^+)\right\} \qquad (5)$$

$$C_4 = -x_c^+ - \frac{a}{q^+}\cos(\omega t_1+\phi-\theta^+)$$

Furthermore, the following continuity conditions for Eqns. (4) exist:

$$x_1(t=t_1) = 0 \qquad \dot{x}_1(t=t_1) = v_0$$
$$x_2(t=t_0) = -x_0 \qquad \dot{x}_2(t=t_0) = 0 \tag{6}$$

Different forms of steady state motion can be obtained for frictionally damped systems. The simplest form is that the system moves contінously and does not come to a stop for a finite period of time. For this type of motion, called non-stop motion, the following condition

$$t_0 = \frac{\pi}{\omega} \tag{7}$$

must be true.
Another possible form would be that the motion comes to a single stop for a finite period of time during one half-cycle, called one-stop motion. For this type of motion not only the velocity but also the acceleration at time t = 0 can be assumed to be equal to zero since the motion is assumed to have stopped for a finite period of time. Hence, for one-stop motion the following condition must hold.

$$x_0(1-\zeta) - x_c = a\cos\phi \tag{8}$$

Other more complicated forms of steady state solutions might be possible and they might involve more than a single stop per half-cycle. Den Hartog (2) showed experimentally that one-stop motion as well as two-stop motion exists. The latter, however, occured only for low frequency excitation which is usually not important for practical purposes. Therefore, types of motion with more than one stop per half-cyle are not considered in the following.

Eqns. (5)-(8) can be used in Eqns. (4) to set up a system of 5 transcendental equations in the unknowns x_o, v_o, t_1, ϕ and t_o. This can be solved, for example, using Newton's method. To assure convergence ζ and x_c were applied step-wise starting from the linear case ($\zeta=x_c=0$) and each intermediate solution was obtained iteratively. For the decision if no-stop or one-stop motion was in effect and hence if Eqn. (7) or (8) had to be taken the acceleration a time t = 0 was monitored. As long as no-stop motion was in effect this acceleration had to be negative for the entire part of the motion considered.
As soon as the frictional damping level was so large that no-stop motion could not exist anymore, the acceleration at time t = 0 changed its sign in the numerical analysis which was the indication that no-stop motion could not exist anymore and hence Enq. (8) had to be taken in the solution process instead of Eqn. (7).

On example each is given for the solution of a particular case of no-stop motion (Fig. 4a) and one-stop motion (Fig. 4b).

Equivalent Linearization

For many practical purposes the use of nonlinear design techniques is not acceptable for design office applications. Particularly during the preliminary design phases the estimates of gross features of the response are often more important than an accurate representation of the response of the system. Therefore, an equivalent linearization of the nonlinear problem is often of great practical importance.
The general concept of equivalent linearization and its application is discussed in detail in the literature.

Here, only a brief summary of two possible linearization procedures will be given for the case of the steady state vibration discussed above.

a) equating work done per cyle

The most popular and well known general linearization method replaces the actual damping force F_d by an equivalent viscous force $c\dot{x}$ in such a way that the equivalent viscous force does the same work per cycle as the actual damping force F_d.

The work done by the actual damping force during one cycle is given by

$$W_{act} = \int F_d \, dx = \int_0^T F_d \dot{x} \, dt \tag{9}$$

The work done by the equivalent viscous force during one cycle is given by

$$W_{eq} = \int_0^T c_{eq} \dot{x} \, \dot{x} \, dt = \int_0^T 2 m \omega_0 \xi_{eq} \dot{x}^2 \, dt \tag{10}$$

Using Eqn. (1) in Eqn. (9) and assuming the resulting motion x to be sinusoidal the following results will be obtained

$$W_{act} = 2 m \xi \omega_0 \, \omega x_0^2 \pi + 4 x_0 F + 2 \zeta k x_0^2$$

$$W_{eq} = 2 m \xi_{eq} \omega_0 \omega x_0^2 \pi$$

Equating these two expression and resolving the resulting equation for ξ_{eq} gives

$$\xi_{eq} = \xi + \frac{2 x_c}{\pi \beta x_0} + \frac{\zeta}{\pi \beta} \tag{11}$$

This result for the equivalent viscous damping ratio still contains the unknown amplitude x_0 in the part of the solution corresponding to the constant Coulomb frictional force.
Using the assumption of sinusoidal motion again this amplitude can be approximated by the following expression

$$x_0 = a \left\{ \left(1-\beta^2\right)^2 + \left(2\beta\xi_{eq}\right)^2 \right\}^{-1/2} \tag{12}$$

With this expression for x_o used in Eqn. (11) the result will be a quadratic equation in ξ_{eq} , which can be solved easily for the equivalent viscous damping ratio.

b) least squares solution

If an exact solution of the problem is available a more elaborate linearization procedure can be used.
A continuous least square approximation between the known exact solution and an equivalent linear solution given in terms of a variable equivalent damping ratio can be carried out. If the least square approximation is limited to one half-cycle of the motion because of the symmetry of the problem then the minimization process is defined by

$$\min_{\xi_{eq}} \int_0^{T/2} \left\{ x_{ex}(t) - x_{eq}(\xi_{eq}, t) \right\}^2 dt$$

Where $x_{ex}(t)$ is the exact solution given above and $x_{eq}(\xi_{eq},t)$ is the equivalent linear solution given as a function of time and the equivalent linear damping ratio. The minimization with respect to the equivalent damping ratio ξ_{eq} is given by

$$2 \int_0^{\frac{\pi}{\omega}} \left\{ x_{ex}(t) - x_{eq}(\xi_{eq}, t) \right\} \frac{\partial x_{eq}}{\partial \xi_{eq}} dt = 0$$

The minimization process can be seen to be equivalent to minimizing the area between the exact and the equivalent linear solution as a function of the equivalent damping ratio ξ_{eq}.

Fig. 5 shows dynamic magnifaction curves and the corresponding equivalent damping ratios found by a least square approximation for a specific case.

Example:

Figs. (3a-c) show hysteresis curves that were published in (1). These curves were experimentally recorded and show the damping forces associated with rubber bearings (3a) and with a sliding friction system (3b and c) with different clearences between the structure and the sliding friction system.
If Eqns. (11) and (12) are used for an equivalent linearization of the problem and if it is assumed that the system is at resonance ($\beta = 1$) the following expression is found for the equivalent damping ratio

$$\xi_{eq} = \frac{\xi + \frac{\zeta}{\pi}}{1 - \frac{4F}{\pi k a}}$$

The viscous damping ratio ξ can be found from the area of the ellipse in Fig. (3a).
The constant friction force F and the linear friction parameter can be approximated from Fig. (3c). For the determination of the linear friction parameter ζ the slope corresponding to 2/3 of the peak value of the damping force is taken rather than the slope corresponding to the peak value itself. This is done since the variation of the damping force corresponding to ζ is linear. If we select that linear variation for the hysteresis loop that has the same area and hence energy as the quadratic variation we arrive at the factor 2/3 for the peak displace-

ment.
With these simple approximations we obtain the following results.

$\xi = 5\%$ $\quad$ $k = 2.88$ (see (1))

$$\zeta = \frac{2/3 \cdot 2.0}{2.88 \cdot 2.33} = 0.20$$

$F = 0.$ (see Fig. 3b)

$F = 1.$ (see Fig. 3c)

The static deflection for the cyclic excitation was assumed to be $a = 1$, which results in a ratio $F/P = 35\%$ for this specific case. Therefore, we obtain for Fig. 3b

$$\xi_{eq} = 0.05 + \frac{0.2}{\pi} = 11\%$$

and for Fig. 3c

$$\xi_{eq} = \frac{0.05 + \frac{0.20}{\pi}}{1 - \frac{4 \cdot 1.0}{\pi \cdot 2.88}} = 20\%$$

Using these results in a spectral analysis for the fundamental frequency ($\omega = 0.64$ Hz) to estimate the peak displacements corresponding to the equivalent damping ratios above and comparing these results with the recorded peak displacements taken from Figs. 3b and c, we obtain

	(x_{max}) recorded	ξ_{eq}	(x_{max}) equivalent
3b	2.77 in	11%	2.54 in
3c	2.33 in	20%	2.08 in

In view of the fact that the damping relation as expressed by the curves in Figs. 3b and c has to be approximated by the relation corresponding to linear viscous damping which is an ellipse (see Fig. 3a), the agreement between the results is surprisingly good.

Conclusion

The exact solution to the nonlinear problem outlined in this paper is certainly not a solution suitable for practical design purposes. It is, however, a very important tool for establishing equivalent linear solutions to be used for practical design purposes. The quality of an equivalent linear solution cannot be judged unless the exact solution is known.

Both linearization methodes discussed were seen to give virtually identical solutions for reasonably small levels of damping ($\xi_{eq} \leqslant 20\%$). These solutions are in very good agreement with the exact solution. For very large levels of damping the considerable additional effort involved with a least square approximation of the problem might be appropriate. Even for those cases, however, the simple solution using the work done per cycle delivers results of satisfactory accuracy for many practical purposes. This is demonstrated for a case of large damping in Figs. 6a and b. These figures show a time history solution for one complete cycle of the exact and the equivalent linear solutions.

References:

(1) Kelly, J.M., Beucke, K.E., "A Friction Damped Base Isolation System with Fail-Safe Characteristics", Earthquake Engineering and Structural Dynamics, Vol. 11, 33-56 (1983).

(2) Den Hartog, J.P., "Forced Vibrations With Combined Coulomb and Viscous Friction", Transactions of the American Society of Mechnical Engineers, Vol. 53, APM-53-9 (1931).

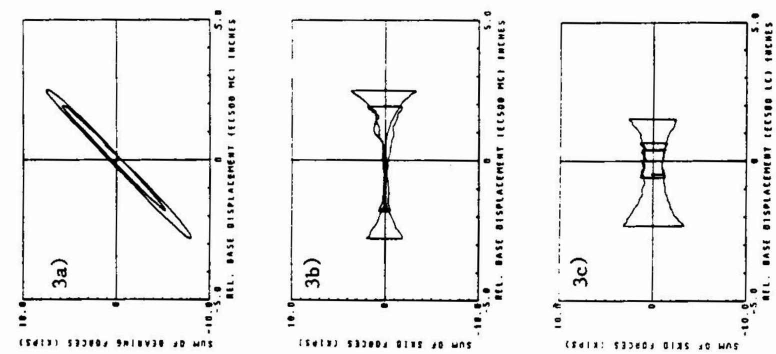


Fig. 3: Force-Displacement Curves

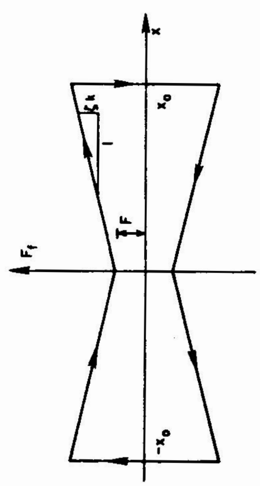


Fig. 1: Frictional Force as a Function of the Displacement

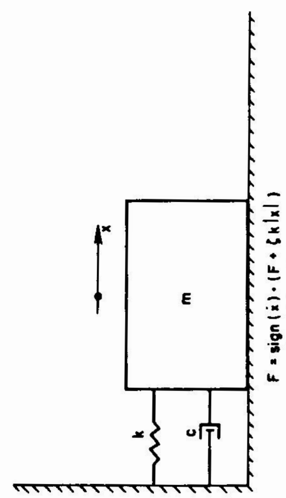
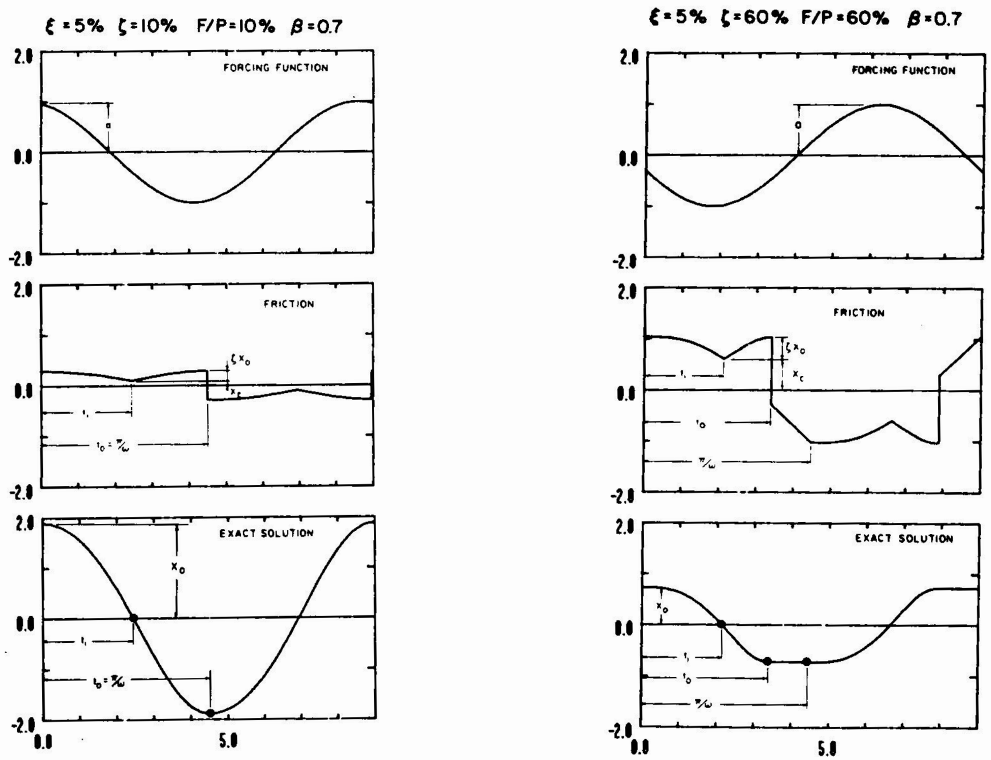


Fig. 2: Single-Degree-of-Freedom Model with Viscous and Friction Damping



a) continuous motion

b) one-stop motion

Fig. 4: Time History of Forcing Function, Frictional Force and Exact Steady-State Solution

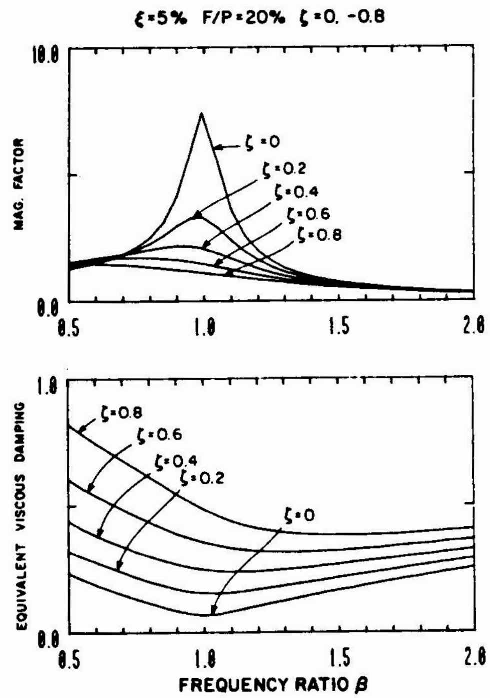


Fig. 5: Amplification Curves and Corresponding Least Square Estimates of Equivalent Viscous Damping for Medium Level of Constant Coulomb Friction

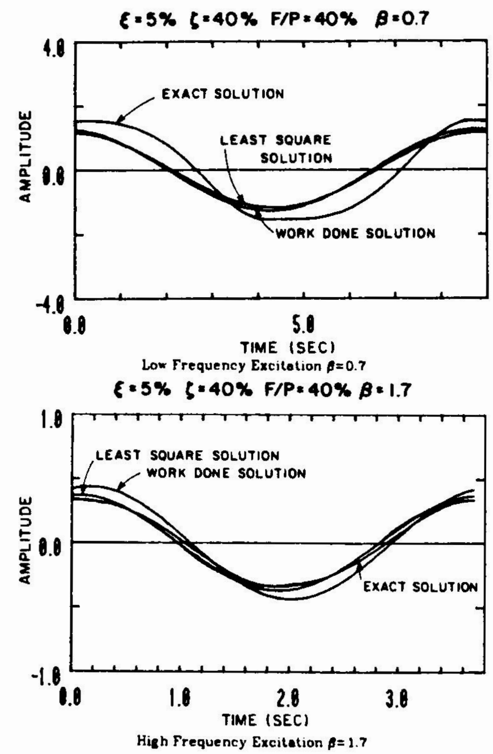


Fig. 6: Time History Solutions for Large Level of General Damping